are shown in Fig. 2, which can be compared with Fig. 1. Note that each of these maneuvers might be suitably approximated by two constant control angles.

Nominal orbit rendezvous maneuvers from both types of J^1 and J^2 evasive maneuvers were computed by solving the TPBVP derived from J^3. Table 2 summarizes the results achieved when considering nominal orbit rendezvous after maximum radius increase maneuvers both with and without terminal orthogonality constraint. Note that the times to rendezvous are significantly greater than the times to evade given in Table 1, and the times for return from a maneuver without terminal constraint are more than twice as great as those from a maneuver with the terminal orthogonality constraint.

The control histories for rendezvous maneuvers from the evasive maneuvers without terminal constraint are shown in Fig. 3, labeled by associated evasion time. After a 2-h evasion, the control angle varies by approximately 12 deg over the maneuver, with the control angle varying linearly for about 3 h of the maneuver. Figure 3 also shows a highly nonlinear behavior of the control angle with time for the rendezvous from the unconstrained 4-h evasive maneuver, which could be more difficult to implement. The control profiles for rendezvous from evasive maneuvers with terminal constraint are shown in Fig. 4, again labeled by evasion time. Note that these control histories depart significantly from those of Fig. 3, and that two constant control angles approximate each maneuver very well.

With the results achieved, the two evasion strategies can be compared using the two tables to estimate the total time required for evasion and rendezvous. Consider the 2- and 4-h radius increase maneuvers of Table 1. The unconstrained 2-h maneuver increases the orbit radius by 26.5 km. From Table 2, 8.49 h are then required to rendezvous with the nominal orbit for a total maneuver time of 10.49 h. On the other hand, with the terminal othogonality constraint, it appears that the orbit radius can be increased by the same 26.5 km, but in 3 h or less. Then, from Table 2, the subsequent rendezvous maneuver appears to require approximately 6 h for a total maneuver time of approximately 9 h. Consequently, time and fuel can be saved by using any available time to shape the evasive maneuver rather than delaying until a maximum radius increase maneuver is required.

Conclusions

Geosynchronous satellite evasive maneuvers using a constant low thrust magnitude and thrust direction control have been modeled in the context of optimal control theory. With evasive maneuvers restricted to the nominal orbit plane, two maneuver objectives were considered. One maximized the change in the orbit radius. The second did the same but required the final radius and velocity vectors to be orthogonal. Numerical results showed that a considerable penalty was paid in the change of robit radius achievable when the terminal constraint was added. Postevasion rendezvous maneuvers with the nominal orbit were also computed. These results showed that rendezvous times can significantly exceed the times allowed for evasive maneuvers. They also showed that a terminal constraint on the evasive maneuver can be used to save significant time and fuel over the complete maneuver sequence when warning time is greater than the minimum required to evade an intercept by some specified change in the orbit radius.

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Optimal Test Procedures for Evaluating Circular Probable Error

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Introduction

EVALUATING the targeting accuracy of a surface-to-surface missile is a rather complicated decision-making process. Especially given the cost of such systems, the decision should be based on the results of a very limited number of firing tests.

In this Note, we propose several tests through which the claimed circular probable error (CEP) (see Ref. 1) of a given missile can be evaluated. Our approach is based on the following observation. Assuming the claimed CEP of a missile is L, then an ideal test procedure is one for which the probability of passing the test is 1 given the actual CEP is less than or equal to L, and the probability of passing the test is 0 given the actual CEP of missile is greater than L. But an actual test will never have such a sharp operating characteristic (OC) curve (see Ref. 2), and we shall devise test procedures whose objective is to minimize the deviation from this ideal characteristic.

Test Procedures

Let us denote the target point at which all missiles are aimed by T and draw n different circles with centers at T and radii R_i such that

$$R_n \geq \cdots \geq R_2 \geq R_1 \geq 0 \tag{1}$$

Let us place a Cartesian coordinate frame on the target and denote the coordinates of a point with reference to this frame by the ordered pair (x,y). We denote the vector with coordinates (x,y) by r and the impact point of the ith missile fired by r_i. Let S_i denote the set of all points whose distance from T is less than or equal to R_i:

$$S_i = (r \mid |r| \leq R_i) \tag{2}$$

Here $|r|$ denotes the Euclidean norm of the vector r. We denote the complement of S_i by S_i', and this is the set of all points outside a circle with radius R_i.

We define a nonsequential test (NST) with (at most) n firing (NSTn) as follows. Fire n missiles aimed at the same target point T. We shall say the missile qualifies and has the claimed CEP if the following n conditions hold simultaneously:

1) At least one missile from n fired should land at a point in S_1.

2) At least two missiles from n fired should land at points in S_2.

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$n-1$) At least $n-1$ missiles from n fired should land at points in S_{n-1}.

n) All n missiles should land at points in S_n.

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Otherwise, the missile does not qualify the test and its CEP is greater than what is claimed.

An important point to note is that sometimes we are not required to fire all n missiles before the NST terminates. For example, if $n = 2$ and the first missile fired lands in S_2', then the missile cannot qualify the test no matter what the outcome of the second firing is, and of course we should not fire the second missile.

Now we define a sequential test (ST) with at most n firing (STn) as follows. Fire the first missile aimed at T.

1) If $r_1 \in S_1$, the missile qualifies and the test terminates. If $r_1 \in S_n'$, then the missile does not qualify and the test terminates. If $r_1 \in S_n \cap S_1'$, the second missile is aimed at T and fired.

2) If $r_2 \in S_2$, the missile qualifies and the test terminates. If $r_2 \in S_n'$, then the missile does not qualify and the test terminates. If $r_2 \in S_n \cap S_2'$, the third missile is aimed at T and fired.

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$n-1$) If $r_{n-1} \in S_{n-1}$, the missile qualifies and the test terminates. If $r_{n-1} \in S_n'$, then the missile does not qualify and the test terminates. If $r_{n-1} \in S_n \cap S_{n-1}'$, the nth missile is aimed at T and fired.

n) If $r_n \in S_n$, the missile qualifies and the test terminates. If $r_n \in S_n'$, the missile does not qualify and the test terminates. Therefore, after at most n firing, the test terminates and the missile either qualifies the test or is rejected.

Mathematical Modeling

In this section, we shall show how to choose the free parameters R_i for the tests defined in the last section. Let us denote by p_i the probability of a fired missile falling inside S_i. Also denote the probability of passing the NST for a given missile system by P, then P for $n = 1$, 2, 3, and 4 is

$$n = 1, \quad P = p_1 \tag{3a}$$

$$n = 2, \quad P = p_1^2 + 2p_1(p_2 - p_1) \tag{3b}$$

$$n = 3, \quad P = p_1^3 + 3p_1^2(p_3 - p_1) + 3p_1(p_2 - p_1)^2 + 6p_1(p_2 - p_1)(p_3 - p_2) \tag{3c}$$

$$n = 4, \quad P = p_1^4 + 4p_1^3(p_4 - p_1) + 6p_1^2(p_3 - p_1)^2 + 12p_1^2(p_3 - p_1)(p_4 - p_3) + 4p_1(p_2 - p_1)^3 + 12p_1(p_2 - p_1)^2(p_4 - p_2) + 12p_1(p_2 - p_1)(p_3 - p_2)^2 + 24p_1(p_2 - p_1)(p_3 - p_2)(p_4 - p_3) \tag{3d}$$

where it is assumed that the missiles fired are independent events. We do not consider the cases $n > 4$ because most probably they are not economically justifiable.

Moreover, if we denote the probability of passing the ST by Q, then

$$Q = p_1 + \sum_{i=2}^{n} p_i \prod_{j=1}^{i-1} (p_n - p_j), \qquad n \geq 2 \tag{4}$$

and $Q = p_1$ for $n = 1$. Note that both NST and ST for $n = 1$ reduce to the same test.

When dealing with an ST, another important quantity in addition to pass probability is the expected number of missiles fired that we shall denote by N. It is simple to show that

$$N = (p_1 + 1 - p_n) + \sum_{i=2}^{n} \left[i(p_i + 1 - p_n) \prod_{j=1}^{i-1} (p_n - p_j) \right], \qquad n \geq 2 \tag{5}$$

and $N = 1$ for $n = 1$. We have to note that all three quantities P, Q, and N are functions of p_1 to p_n, and as we shall show, p_1 to p_n are themselves functions of radii R_1 to R_n and the error parameters of the missile.

Also, let us assume that the errors of the missile in range and cross range are independent and have Gaussian distribution with zero mean and equal standard deviation σ, then it is well known that the probability of a missile falling inside a circle with radius R_i is[3]

$$p_i = P(|r| \leq R_i) = 1 - e^{-R_i^2/(2\sigma^2)} \tag{6}$$

Also, the CEP of a missile by definition is the radius of a circle within which a missile will hit with a probability of 0.5.[1] Using this definition and Eq. (6), we have CEP $= 1.1774\sigma$. For normalizing the problem, in what will follow we assume that the claimed CEP of the missile in 1.1774 units of distance, or equivalently the claimed standard deviation σ is unity. Now substituting Eq. (6) for p_i in Eq. (3), it is clear that the quantity P is a function of R_1 to R_n and σ; hence, we shall use the notation $P(R_1, \ldots, R_n, \sigma)$. Similar arguments hold for Q and N, and hence we use $Q(R_1, \ldots, R_n, \sigma)$ and $N(R_1, \ldots, R_n, \sigma)$.

Moreover, for a given set of radii R_1 to R_n, P is a monotonically decreasing function of the actual standard deviation σ. Using this observation and noting that our objective is to minimize the consumer risk (type II error[2]) for a prespecified producer risk (type I error[2]), we formulate the following optimization problem for finding the radii R_1 to R_n in NSTn:

$$\min_{R_1, \ldots, R_n} P(R_1, \ldots, R_n, \sigma_0) \tag{7}$$

subject to $P(R_1, \ldots, R_n, 1) = P_0$. Here P_0 is the prespecified probability of passing the test, assuming the actual standard deviation of the missile σ is equal to that claimed (i.e., unity). The producer risk is defined to be $1 - P_0$, and we choose $P_0 = 0.95$. Also σ_0 is the actual standard deviation of the missile for which we want to minimize the pass probability. For a given set of radii, we shall call $P(R_1, \ldots, R_n, \sigma_0)$ the consumer risk and set $\sigma_0 = 2$.

For computing the radii R_1 to R_n in the sequential test, we minimize a combination of Q and N:

$$\min_{R_1, \ldots, R_n} Q(R_1, \ldots, R_n, \sigma_0) + \mu N(R_1, \ldots, R_n, 1) \tag{8}$$

subject to $Q(R_1, \ldots, R_n, 1) = P_0$. Here, the same arguments as stated in NSTn for P_0 and σ_0 hold. Also, Q and N are always non-negative quantities, and the parameter $\mu > 0$ is a tradeoff factor for weighting the pass probability against the expected number of missiles fired.

Optimization Results

For solving the optimization problems of the last section, we used a gradient projection algorithm.[4] We first stated the results for NST. The optimal radii were found for $P_0 = 0.95$, and these values are listed in Table 1. The last column in this table lists the consumer risk that is the value of $P(R_1, \ldots, R_n, 2)$ for optimal radii. Note that for $n = 1$ there is only one radius (R_1) to be determined in both NST1 and ST1. By specifying P_0, R_1 will be fixed and in this case there is nothing to optimize for.

The OC curves for the tests with radii given in Table 1 are plotted in Fig. 1. It is clear from this figure that the OC curves for $n = 3$ and 4 have rather a sharp transition characteristic and

Table 1 Optimal radii for NST ($P_0 = 0.95$, $\sigma_0 = 2$)

n	R_1	R_2	R_3	R_4	P
1	2.4477	—	—	—	0.5271
2	2.0000	2.8283	—	—	0.3426
3	1.7741	2.2132	3.0520	—	0.2262
4	1.6287	1.9329	2.3654	3.2096	0.1496

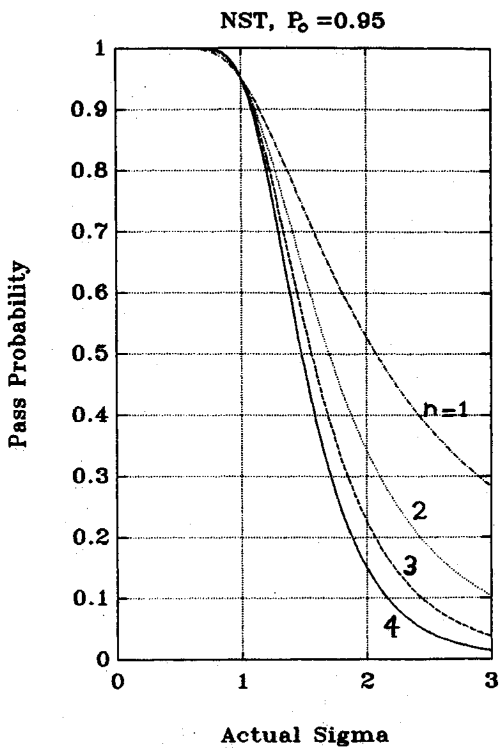

Fig. 1 Operating characteristic curves for nonsequential tests.

Table 2 Optimal radii for ST2 ($P_0 = 0.95$, $\sigma_0 = 2$)

μ	R_1	R_2	Q	N
0.1	1.1480	2.6040	0.3917	1.48
0.2	1.5772	2.5367	0.4250	1.25
0.3	1.8090	2.5046	0.4486	1.15

Table 3 Optimal radii for ST3 ($P_0 = 0.95$, $\sigma_0 = 2$)

μ	R_1	R_2	R_3	Q	N
0.050	0.3420	1.0646	2.7865	0.2789	2.43
0.075	0.8409	1.1649	2.7169	0.3047	2.00
0.100	1.0745	1.2396	2.6698	0.3254	1.77

Table 4 Optimal radii for ST4 ($P_0 = 0.95$, $\sigma_0 = 2$)

μ	R_1	R_2	R_3	R_4	Q	N
0.043	0.0001	0.8629	1.1072	2.8606	0.2199	2.99
0.075	0.7616	1.1257	1.2120	2.7616	0.2624	2.26
0.094	0.9712	1.1912	1.2504	2.7161	0.2847	2.00

their performance is quite acceptable given the small number of firings required by each of these tests.

The optimization results for STn were computed for different values of μ. Results for $n = 2$, 3, and 4 are listed in Tables 2, 3, and 4, respectively. Symbols Q and N in the last two columns of these tables represent $Q(R_1, \ldots, R_n, 2)$ and $N(R_1, \ldots, R_n, 1)$, respectively.

We also chose some of the weightings μ listed in these tables based on the resulting N. For example, in ST3 (Table 3 with $\mu = 0.075$), the resulting N is approximately 2. Of course, if our purpose was to limit N to only integer values, then we could add the constraint $N(R_1, \ldots, R_n, 1) = N_0$ to the optimization problem formulated in Eq. (8). It may be desirable in some cases to limit N_0 to only integer values; however, by changing μ in Eq. (8) it is possible to concentrate on minimizing the consumer risk that is the main quantity in our problem, and one should not put too much emphasis on N_0 being an integer value.

Some interesting observations about the results in these tables can be made. For example, the performance of ST3 with $\mu = 0.075$ in Table 3 is superior to NST2 given in Table 1, and the expected number of missiles for ST3 in this case is two, the same as the number of missiles required for NST2. Of course, there is always a possibility that ST3 requires all three missiles to be fired, and in this case its performance is inferior to NST3.

Finally, if the claimed σ is not unity but some other number M, then the radii given in Tables 1–4 can be scaled by the factor M. Also, if for a particular missile the claimed standard deviation of error in range (σ_x) and in cross range (σ_y) are not equal, the tests suggested here can still be used. For this, define $a_i = R_i\sigma_x$ and $b_i = R_i\sigma_y$ where R_i are the radii given in Tables 1–4, and redefine the sets S_i as the interior of ellipses with semiaxis a_i and b_i as follows:

$$S_i = \{r \mid x^2/(a_i^2) + y^2/(b_i^2) \leq 1\} \tag{9}$$

Conclusion

In this Note, the problem of confirming or rejecting the claimed CEP of a missile was discussed, and several test procedures were suggested for this purpose. The quantities such as the producer risk, the consumer risk, and the expected number of missiles fired were considered in deriving the test parameters. Some of the tests suggested here have very good performance; however, it should be emphasized that our problem statistically concerns the evaluation of a claimed standard deviation, and it is very difficult to evaluate this quantity using a very small number of sample points.

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